

# Approximation by Genuine $q$ -Bernstein-Durrmeyer Polynomials in Compact Disks in the case $q > 1$

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## Abstract

This paper deals with approximating properties of the newly defined  $q$ -generalization of the genuine Bernstein-Durrmeyer polynomials in the case  $q > 1$ , which are no longer positive linear operators on  $C[0, 1]$ . Quantitative estimates of the convergence, the Voronovskaja type theorem and saturation of convergence for complex genuine  $q$ -Bernstein-Durrmeyer polynomials attached to analytic functions in compact disks are given. In particular, it is proved that for functions analytic in  $\{z \in \mathbb{C} : |z| < R\}$ ,  $R > q$ , the rate of approximation by the genuine  $q$ -Bernstein-Durrmeyer polynomials ( $q > 1$ ) is of order  $q^{-n}$  versus  $1/n$  for the classical genuine Bernstein-Durrmeyer polynomials. We give explicit formulas of Voronovskaja type for the genuine  $q$ -Bernstein-Durrmeyer for  $q > 1$ .

## 1 Introduction

In several recent papers, convergence properties of complex  $q$ -Bernstein polynomials, proposed by Phillips [3], attached to an analytic function  $f$  in closed disks, were intensively studied. Ostrovska [17], [18], and Wang and Wu [21], [22] have investigated convergence properties of  $B_{n,q}$  in the case  $q > 1$ . In the case  $q > 1$ , the  $q$ -Bernstein polynomials are no longer positive operators, however, for a function analytic in a disc  $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$ ,  $R > q$ , it was proved in [17] that the rate of convergence of  $\{B_{n,q}(f; z)\}$  to  $f(z)$  has the order  $q^{-n}$  (versus  $1/n$  for the classical Bernstein polynomials). Moreover, Ostrovska [18] obtained Voronovskaja type theorem for monomials. If  $q \geq 1$  then qualitative Voronovskaja-type and saturation results for complex  $q$ -Bernstein polynomials were obtained in Wang-Wu [21]. Wu [22] studied saturation of convergence on the interval  $[0, 1]$  for the  $q$ -Bernstein polynomials of a continuous function  $f$  for arbitrary fixed  $q > 1$ .

Genuine Bernstein-Durrmeyer operators were first considered by Chen [10] and Goodman and Sharma [15] around 1987. In recent years, the genuine Bernstein-Durrmeyer operators have been investigated intensively by a number of authors. Among the many articles written on the genuine Bernstein-Durrmeyer operators, we mention here only the ones by Gonska and etc [13], by Parvanov and Popov [5], by Sauer [7], by Waldron [8], and the book of Páltánea [9].

On the other hand, Gal [4] obtained quantitative estimates of the convergence and of the Voronovskaja theorem in compact disks, for the complex genuine Bernstein-Durrmeyer polynomials attached to analytic functions. Besides, in other very recent papers, similar studies were done for complex Bernstein-Durrmeyer operators in Anastassiou-Gal [16], for complex Bernstein-Durrmeyer operators based on Jacobi weights in Gal [23], for complex genuine Bernstein-Durrmeyer operator in Gal [24], for complex  $q$ -genuine Bernstein-Durrmeyer operator in Mahmudov [25] and for other kinds of complex Durrmeyer operators in Mahmudov [26] and Gal-Gupta-Mahmudov [28]. Also, for the case  $q > 1$ , exact quantitative estimates and quantitative Voronovskaja-type results for complex  $q$ -Lorentz polynomials,  $q$ -Stancu polynomials,  $q$ -Stancu-Faber polynomials,  $q$ -Bernstein-Faber polynomials,  $q$ -Kantorovich polynomials,  $q$ -Szász-Mirakjan operators obtained by different researchers are collected in the recent book of Gal [29].

In this paper we define the genuine  $q$ -Bernstein-Durrmeyer polynomials for  $q > 1$ . Note that similar to the  $q$ -Bernstein operators the genuine  $q$ -Bernstein-Durrmeyer operators in the case  $q > 1$  are not positive operators on  $C[0, 1]$ . The lack of positivity makes the investigation of convergence in the case  $q > 1$  essentially more difficult than that for  $0 < q < 1$ . We present upper estimates in approximation and we prove the Voronovskaja type convergence theorem in compact disks in  $\mathbb{C}$ , centered at origin, with quantitative estimate of this convergence. These results allow us to obtain the exact degrees of approximation

by complex genuine  $q$ -Bernstein-Durrmeyer polynomials. Our results show that approximation properties of the complex genuine  $q$ -Bernstein-Durrmeyer polynomials are better than approximation properties of the complex Bernstein-Durrmeyer polynomials considered in [4].

## 2 Formulation

Let  $q > 0$ . For any  $n \in \mathbb{N} \cup \{0\}$ , the  $q$ -integer  $[n]_q$  is defined by

$$[n]_q := 1 + q + \dots + q^{n-1}, \quad [0]_q := 0;$$

and the  $q$ -factorial  $[n]_q!$  by

$$[n]_q! := [1]_q [2]_q \dots [n]_q, \quad [0]_q! := 1.$$

For integers  $0 \leq k \leq n$ , the  $q$ -binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

For  $q = 1$  we obviously get  $[n]_q = n$ ,  $[n]_q! = n!$ ,  $\begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$ . Moreover

$$(1-z)_q^n := \prod_{s=0}^{n-1} (1 - q^s z), \quad p_{n,k}(q; z) := \begin{bmatrix} n \\ k \end{bmatrix}_q z^k (1-z)_q^{n-k}, \quad z \in \mathbb{C}.$$

For fixed  $q > 0$ ,  $q \neq 1$ , we denote the  $q$ -derivative  $D_q f(z)$  of  $f$  by

$$D_q f(z) = \begin{cases} \frac{f(qz) - f(z)}{(q-1)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

The  $q$ -analogue of integration in the interval  $[0, A]$  (see [1]) is defined by

$$\int_0^A f(t) d_q t := A(1-q) \sum_{n=0}^{\infty} f(Aq^n) q^n, \quad 0 < q < 1.$$

Let  $\mathbb{D}_R$  be a disc  $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$  in the complex plane  $\mathbb{C}$ . Denote by  $H(\mathbb{D}_R)$  the space of all analytic functions on  $\mathbb{D}_R$ . For  $f \in H(\mathbb{D}_R)$  we assume that  $f(z) = \sum_{m=0}^{\infty} a_m z^m$ .

**Definition 1** For  $f : [0, 1] \rightarrow \mathbb{C}$ , the genuine  $q$ -Bernstein-Durrmeyer operator is defined as follows:

$$U_{n,q}(f; z) := \begin{cases} f(0)p_{n,0}(q; z) + f(1)p_{n,n}(q; z) \\ + [n-1]_q \sum_{k=1}^{n-1} q^{1-k} p_{n,k}(q; z) \int_0^1 p_{n-2,k-1}(q; qt) f(t) d_q t, & 0 < q < 1, \\ f(0)p_{n,0}(z) + f(1)p_{n,n}(z) + (n-1) \sum_{k=1}^{n-1} p_{n,k}(z) \int_0^1 p_{n-2,k-1}(t) f(t) dt, & q = 1, \\ f(0)p_{n,0}(q; z) + f(1)p_{n,n}(q; z) \\ + [n-1]_{q^{-1}} \sum_{k=1}^{n-1} q^{k-1} p_{n,k}(q; z) \int_0^1 p_{n-2,k-1}(q^{-1}; q^{-1}t) f(q^{k-n}t) d_{q^{-1}} t, & q > 1, \end{cases} \quad (1)$$

where for  $n = 1$  the sum is empty, i.e., equal to 0.

$U_{n,q}(f; z)$  are linear operators reproducing linear functions and interpolating every function  $f \in C[0, 1]$  at 0 and 1. The genuine  $q$ -Bernstein-Durrmeyer operators are positive operators on  $C[0, 1]$  for  $0 < q \leq 1$ , and they are not positive for  $q > 1$ . As a consequence, the cases  $0 < q \leq 1$  and  $q > 1$  are

not similar to each other regarding the convergence. For  $q \rightarrow 1^-$  and  $q \rightarrow 1^+$  we recapture the classical ( $q = 1$ ) genuine Bernstein-Durrmeyer polynomials.

We start with the following quantitative estimates of the convergence for complex  $q$ -Bernstein-Durrmeyer polynomials attached to an analytic function in a disk of radius  $R > 1$  and center 0.

**Theorem 2** *Let  $f \in H(\mathbb{D}_R)$ ,  $1 \leq r < \frac{R}{q}$  and  $q > 1$ . Then for all  $|z| \leq r$  we have*

$$|U_{n,q}(f; z) - f(z)| \leq \frac{r(1+r)}{[n+1]_q} \sum_{m=2}^{\infty} |a_m| m(m-1) q^{m-2} r^{m-2}.$$

Theorem 2 says that for functions analytic in  $\mathbb{D}_R$ ,  $R > q$ , the rate of approximation by the genuine  $q$ -Bernstein-Durrmeyer polynomials ( $q > 1$ ) is of order  $q^{-n}$  versus  $1/n$  for the classical genuine Bernstein-Durrmeyer polynomials, see [4].

The Voronovskaja theorem for the real case with a quantitative estimate is obtained by Gonska, Pişul and Raşa [14] in the following form:

$$\left| U_n(f; x) - f(x) - \frac{x(1-x)}{n+1} f''(x) \right| \leq \frac{x(1-x)}{n+1} \omega\left(f''; \frac{2}{3\sqrt{n+3}}\right),$$

for all  $n \in \mathbb{N}$ ,  $0 \leq x \leq 1$ . For the complex genuine  $q$ -Bernstein-Durrmeyer ( $0 < q \leq 1$ ) a quantitative estimate is obtained by Gal [4] ( $q = 1$ ) and Mahmudov [25] ( $0 < q < 1$ ) in the following form:

$$\left| U_{n,q}(f; z) - f(z) - \frac{z(1-z)}{[n+1]_q} f''(z) \right| \leq \frac{M_{r,f}}{[n]_q^2}, \quad 0 < q \leq 1,$$

for all  $n \in \mathbb{N}$ ,  $|z| \leq r$ .

To formulate and prove the Voronovskaja type theorem with a quantitative estimate in the case  $q > 1$  we introduce a function  $L_q(f; z)$ .

Let  $R > q \geq 1$  and let  $f \in H(\mathbb{D}_R)$ . For  $|z| < R/q^2$ , we define

$$L_q(f; z) := \frac{(1-z)q(D_q f(z) - D_{q^{-1}} f(z))}{q-1} \quad \text{for } q > 1 \quad (2)$$

and for  $0 < q \leq 1$ ,

$$L_q(f; z) = L_1(f; z) := f''(z) z(1-z).$$

The next theorem gives Voronovskaja type result in compact disks, for complex  $q$ -Bernstein-Durrmeyer polynomials attached to an analytic function in  $\mathbb{D}_R$ ,  $R > q^2 > 1$  and center 0 in terms of the function  $L_q(f; z)$ .

**Theorem 3** *Let  $f \in H(\mathbb{D}_R)$ ,  $1 \leq r < \frac{R}{q^2}$  and  $q > 1$ . The following Voronovskaja-type result holds*

$$\left| U_{n,q}(f; z) - f(z) - \frac{1}{[n+1]_q} L_q(f; z) \right| \leq \frac{4r^2(1+r)^2}{[n+1]_q^2} \sum_{m=3}^{\infty} |a_m| (m-1)^2 (m-2)^2 (q^2 r)^{m-2}.$$

for all  $n \in \mathbb{N}$ ,  $|z| \leq r$ .

Now we are in position to prove that the order of approximation in Theorem 2 is exactly  $q^{-n}$  versus  $1/n$  for the classical genuine Bernstein-Durrmeyer polynomials, see [4].

**Theorem 4** *Let  $1 < q < R$ ,  $1 \leq r < \frac{R}{q^2}$  and  $f \in H(\mathbb{D}_R)$ . If  $f$  is not a polynomial of degree  $\leq 1$ , the estimate*

$$\|U_{n,q}(f) - f\|_r \geq \frac{1}{[n+1]_q} C_{r,q}(f), \quad n \in \mathbb{N},$$

holds, where the constant  $C_{r,q}(f)$  depends on  $f$ ,  $q$  and  $r$  but is independent of  $n$ .

From Theorem 3 we conclude that for  $q > 1$ ,  $[n+1]_q (U_{n,q}(f; z) - f(z)) \rightarrow L_q(f; z)$  in  $H(\mathbb{D}_{R/q})$  and therefore,  $L_q(f; z) \in H(\mathbb{D}_{R/q})$ . Furthermore, we have the following saturation of convergence for the genuine  $q$ -Bernstein-Durrmeyer polynomials for fixed  $q > 1$ .

**Theorem 5** *Let  $1 < q < R$ ,  $1 \leq r < \frac{R}{q^2}$ . If a function  $f$  is analytic in the disc  $\mathbb{D}_{R/q}$ , then  $|U_{n,q}(f; z) - f(z)| = o(q^{-n})$  for infinite number of points having an accumulation point on  $\mathbb{D}_{R/q}$  if and only if  $f$  is linear.*

The next theorem shows that  $L_q(f; z)$ ,  $q \geq 1$ , is continuous in the parameter  $q$  for  $f \in H(\mathbb{D}_R)$ ,  $R > 1$ .

**Theorem 6** *Let  $R > 1$  and  $f \in H(\mathbb{D}_R)$ . Then for any  $r$ ,  $0 < r < R$ ,*

$$\lim_{q \rightarrow 1+} L_q(f; z) = L_1(f; z)$$

*uniformly on  $\mathbb{D}_R$ .*

### 3 Auxiliary results

The  $q$ -analogue of Beta function for  $0 < q < 1$  (see [1]) is defined as

$$B_q(m, n) = \int_0^1 t^{m-1} (1 - qt)_q^{n-1} d_q t, \quad m, n > 0, \quad 0 < q < 1.$$

Since we consider the case  $q > 1$ , we need to use  $B_{q^{-1}}(m, n)$ :

$$B_{q^{-1}}(m, n) = \int_0^1 t^{m-1} (1 - q^{-1}t)_{q^{-1}}^{n-1} d_{q^{-1}} t, \quad m, n > 0, \quad 0 < q^{-1} < 1.$$

Also, it is known that

$$B_{q^{-1}}(m, n) = \frac{[m-1]_{q^{-1}}! [n-1]_{q^{-1}}!}{[m+n-1]_{q^{-1}}!}, \quad 0 < q^{-1} < 1.$$

For  $m = 0, 1, \dots$ , we have

$$\begin{aligned} & [n-1]_{q^{-1}} q^{k-1} \int_0^1 t^m p_{n-2,k-1}(q^{-1}; q^{-1}t) d_{q^{-1}} t \\ &= [n-1]_{q^{-1}} \left[ \begin{matrix} n-2 \\ k-1 \end{matrix} \right]_{q^{-1}} q^{m(k-n)} \int_0^1 t^{k+m-1} (1 - q^{-1}t)_{q^{-1}}^{n-k-1} d_{q^{-1}} t \\ &= q^{m(k-n)} \frac{[n-1]_{q^{-1}}!}{[k-1]_{q^{-1}}! [n-k-1]_{q^{-1}}!} B_{q^{-1}}(k+m, n-k) \\ &= q^{m(k-n)} \frac{[n-1]_{q^{-1}}!}{[k-1]_{q^{-1}}! [n-k-1]_{q^{-1}}!} \frac{[k+m-1]_{q^{-1}}! [n-k-1]_{q^{-1}}!}{[k+m+n-k-1]_{q^{-1}}!} \\ &= \frac{[n-1]_q! [k+m-1]_q!}{[k-1]_q! [n+m-1]_q!} = \frac{[k+m-1]_q \dots [k]_q}{[n+m-1]_q \dots [n]_q}. \end{aligned}$$

Thus, we get the following formula for  $U_{n,q}(e_m; z)$ :

$$\begin{aligned} U_{n,q}(e_m; z) &= f(0) p_{n,0}(q; z) + f(1) p_{n,n}(q; z) \\ &\quad + [n-1]_{q^{-1}} \sum_{k=1}^{n-1} p_{n,k}(q; z) \int_0^1 p_{n-2,k-1}(q^{-1}; q^{-1}t) f(q^{k-n}t) d_{q^{-1}} t \\ &= z^n + \sum_{k=1}^{n-1} p_{n,k}(q; z) \frac{[k+m-1]_q \dots [k]_q}{[n+m-1]_q \dots [n]_q}. \end{aligned} \tag{3}$$

Note for  $m = 0, 1, 2$  we have

$$U_{n,q}(e_0; z) = 1, \quad U_{n,q}(e_1; z) = z, \quad U_{n,q}(e_2; z) = z^2 + \frac{(1+q)z(1-z)}{[n+1]}.$$

**Lemma 7**  $U_{n,q}(e_m; z)$  is a polynomial of degree less than or equal to  $\min(m, n)$  and

$$U_{n,q}(e_m; z) = \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^m S_q(m, s) [n]_q^s B_{n,q}(e_s; z).$$

**Proof.** From (3) it follows that

$$\begin{aligned} U_{n,q}(e_m; z) &= \sum_{k=1}^n p_{n,k}(q; z) \frac{[k+m-1]_q \cdots [k]_q}{[n+m-1]_q \cdots [n]_q} \\ &= \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{k=1}^n [k]_q [k+1]_q \cdots [k+m-1]_q p_{n,k}(q; z). \end{aligned}$$

Now using

$$[k]_q [k+1]_q \cdots [k+m-1]_q = \prod_{s=0}^{m-1} (q^s [k]_q + [s]_q) = \sum_{s=1}^m S_q(m, s) [k]_q^s, \quad (4)$$

where  $S_q(m, s) > 0$ ,  $s = 1, 2, \dots, m$ , are the constants independent of  $k$ , we get

$$\begin{aligned} U_{n,q}(e_m; z) &= \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{k=0}^n \sum_{s=1}^m S_q(m, s) [k]_q^s p_{n,k}(q; z) \\ &= \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^m S_q(m, s) [n]_q^s B_{n,q}(e_s; z), \end{aligned}$$

Since  $B_{n,q}(e_s; z)$  is a polynomial of degree less than or equal to  $\min(s, n)$  and  $S_q(m, s) > 0$ ,  $s = 1, 2, \dots, m$ , it follows that  $U_{n,q}(e_m; z)$  is a polynomial of degree less than or equal to  $\min(m, n)$ . ■

**Lemma 8** The numbers  $S_q(m, s)$ ,  $(m, s) \in (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$ , given by (4) enjoy the following properties

$$\begin{aligned} S_q(0, 0) &= 1, \quad S_q(m, 0) = 0, \quad m \in \mathbb{N}, \\ S_q(m+1, s) &= [m]_q S_q(m, s) + q^m S_q(m, s-1), \quad m \in \mathbb{N}_0, \quad s \in \mathbb{N}, \\ S_q(m+1, m+1) &= q^m S_q(m, m), \quad S_q(m, s) = 0 \quad \text{for } s > m. \end{aligned}$$

Also, the following lemma holds.

**Lemma 9** For all  $m, n \in \mathbb{N}$  the identity

$$\frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^m S_q(m, s) [n]_q^s = 1,$$

holds.

**Proof.** It follows from end points interpolation property of  $U_{n,q}(e_m; z)$  and  $B_{n,q}(e_s; z)$ . Indeed

$$1 = U_{n,q}(e_m; 1) = \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^m S_q(m, s) [n]_q^s B_{n,q}(e_s; 1) = \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^m S_q(m, s) [n]_q^s.$$

■

Lemma 9 implies that for all  $m, n \in \mathbb{N}$  and  $|z| \leq r$  we have

$$\begin{aligned} |U_{n,q}(e_m; z)| &\leq \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^m S_q(m, s) [n]_q^s |B_{n,q}(e_s; z)| \\ &\leq \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{s=1}^m S_q(m, s) [n]_q^s r^s \leq r^m. \end{aligned} \quad (5)$$

For our purpose first we need a recurrence formula for  $U_{n,q}(e_m; z)$ .

**Lemma 10** *For all  $m, n \in \mathbb{N} \cup \{0\}$  and  $z \in \mathbb{C}$  we have*

$$U_{n,q}(e_{m+1}; z) = \frac{q^m z (1-z)}{[n+m]_q} D_q U_{n,q}(e_m; z) + \frac{q^m [n]_q z + [m]_q}{[n+m]_q} U_{n,q}(e_m; z). \quad (6)$$

**Proof.** By simple calculation we obtain (see [27])

$$z(1-z) D_q(p_{n,k}(q; z)) = ([k]_q - [n]_q z) p_{n,k}(q; z),$$

and

$$\begin{aligned} U_{n,q}(e_m; z) &= z^n + \sum_{k=1}^{n-1} p_{n,k}(q; z) \frac{[k+m-1]_q \dots [k]_q}{[n+m-1]_q \dots [n]_q} \\ &= z^n + \sum_{k=1}^{n-1} p_{n,k}(q; z) I_{k,m}, \\ I_{k,m} &:= \frac{[k+m-1]_q \dots [k]_q}{[n+m-1]_q \dots [n]_q}. \end{aligned}$$

It follows that

$$\begin{aligned} &z(1-z) D_q U_{n,q}(e_m; z) \\ &= [n]_q z(1-z) z^{n-1} + \sum_{k=1}^{n-1} ([k]_q - [n]_q z) p_{n,k}(q; z) I_{k,m} \\ &= [n]_q z^n + \sum_{k=1}^{n-1} [k]_q p_{n,k}(q; z) I_{k,m} - [n]_q z \sum_{k=1}^{n-1} p_{n,k}(q; z) I_{k,m} - [n]_q z^{n+1} \\ &= [n]_q z^n + \sum_{k=1}^{n-1} [k]_q p_{n,k}(q; z) I_{k,m} - z [n]_q U_{n,q}(e_m; z) \\ &= [n]_q z^n + q^{-m} \sum_{k=1}^{n-1} p_{n,k}(q; z) \left( q^m [k]_q + [m]_q - [m]_q \right) I_{k,m} - z [n]_q U_{n,q}(e_m; z) \\ &= [n]_q z^n + q^{-m} \sum_{k=1}^{n-1} p_{n,k}(q; z) \left( q^m [k]_q + [m]_q - [m]_q \right) I_{k,m} - z [n]_q U_{n,q}(e_m; z) \\ &= q^{-m} \left( q^m [n]_q + [m]_q - [m]_q \right) z^n + q^{-m} [n+m]_q \sum_{k=1}^{n-1} p_{n,k}(q; z) I_{k,m+1} \\ &\quad - q^{-m} [m]_q \sum_{k=1}^{n-1} p_{n,k}(q; z) I_{k,m} - z [n]_q U_{n,q}(e_m; z) \\ &= q^{-m} [n+m]_q U_{n,q}(e_{m+1}; z) - q^{-m} [m]_q U_{n,q}(e_m; z) - z [n]_q U_{n,q}(e_m; z), \end{aligned} \quad (7)$$

which implies the recurrence in the statement. ■

Let

$$\Theta_{n,m}(q; z) := U_{n,q}(e_m; z) - z^m - \frac{1}{[n+1]_q} \left( q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) z^{m-1} (1-z).$$

Using the recurrence formula (6) we prove two more recurrence formulas.

**Lemma 11** *For all  $m, n \in \mathbb{N}$  and  $z \in \mathbb{C}$  we have*

$$\begin{aligned} U_{n,q}(e_m; z) - z^m &= \frac{q^{m-1}z(1-z)}{[n+m-1]_q} D_q U_{n,q}(e_{m-1}; z) \\ &\quad + \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q} (U_{n,q}(e_{m-1}; z) - z^{m-1}) + \frac{[m-1]_q}{[n+m-1]_q} (1-z) z^{m-1}, \end{aligned} \quad (8)$$

$$\begin{aligned} \Theta_{n,m}(q; z) &= \frac{q^{m-1}z(1-z)}{[n+m-1]_q} D_q (U_{n,q}(e_{m-1}; z) - z^{m-1}) \\ &\quad + \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q} \Theta_{n,m-1}(q; z) + R_{n,m}(q; z), \end{aligned} \quad (9)$$

where

$$R_{n,m}(q; z) = \frac{[m-1]_q}{[n+m-1]_q [n+1]_q} \left[ (1 + q^{m-1}) + \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right) (z+1) \right] z^{m-2} (1-z). \quad (10)$$

**Proof.** From the recurrence formula in Lemma 10, for all  $m \geq 2$  we get

$$\begin{aligned} U_{n,q}(e_m; z) - z^m &= \frac{q^{m-1}z(1-z)}{[n+m-1]_q} D_q U_{n,q}(e_{m-1}; z) + \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q} (U_{n,q}(e_{m-1}; z) - z^{m-1}) \\ &\quad + \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q} z^{m-1} - z^m \\ &= \frac{q^{m-1}z(1-z)}{[n+m-1]_q} D_q U_{n,q}(e_{m-1}; z) + \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q} (U_{n,q}(e_{m-1}; z) - z^{m-1}) \\ &\quad + \frac{[m-1]_q}{[n+m-1]_q} (1-z) z^{m-1}, \end{aligned}$$

and

$$\begin{aligned} U_{n,q}(e_m; z) - z^m &- \frac{1}{[n+1]_q} \left( q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) z^{m-1} (1-z) \\ &= \frac{q^{m-1}z(1-z)}{[n+m-1]_q} D_q (U_{n,q}(e_{m-1}; z) - z^{m-1}) \\ &\quad + \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q} \left( U_{n,q}(e_m; z) - z^{m-1} - \frac{1}{[n+1]_q} \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right) z^{m-2} (1-z) \right) \\ &\quad + R_{n,m}(q; z), \end{aligned}$$

where

$$\begin{aligned}
R_{n,m}(q; z) &= \frac{[m-1]_q}{[n+m-1]_q} (1-z) z^{m-1} - \frac{1}{[n+1]_q} \left( q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) z^{m-1} (1-z) \\
&\quad + \frac{q^{m-1} [m-1]_q}{[n+m-1]_q} (1-z) z^{m-1} \\
&\quad + \frac{q^{m-1} [n]_q z + [m-1]_q}{[n+m-1]_q} \frac{1}{[n+1]_q} \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right) z^{m-2} (1-z) \\
&:= T_{n',m}(q) z^{m-1} (1-z) + \frac{[m-1]_q}{[n+m-1]_q [n+1]_q} \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right) z^{m-2} (1-z).
\end{aligned}$$

Again by simple calculation we obtain

$$\begin{aligned}
T_{n,m}(q) &= \frac{[m-1]_q}{[n+m-1]_q} - \frac{1}{[n+1]_q} \left( q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) \\
&\quad + \frac{q^{m-1} [m-1]_q}{[n+m-1]_q} + \frac{q^{m-1} [n]_q}{[n+m-1]_q} \frac{1}{[n+1]_q} \left( q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) \\
&\quad - \frac{q^{m-1} [n]_q}{[n+m-1]_q} \frac{1}{[n+1]_q} \left( q [m-1]_q + [m-1]_{q^{-1}} \right) \\
&= \left( \frac{[m-1]_q}{[n+m-1]_q} + \frac{q^{m-1} [m-1]_q}{[n+m-1]_q} - \frac{q^{m-1} [n]_q}{[n+m-1]_q} \frac{1}{[n+1]_q} \left( q [m-1]_q + [m-1]_{q^{-1}} \right) \right) \\
&\quad + \left( \frac{q^{m-1} [n]_q}{q^{m-1} [n]_q + [m-1]_q} - 1 \right) \frac{1}{[n+1]_q} \left( q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) \\
&:= T_{n,m}^1(q) + T_{n,m}^2(q),
\end{aligned}$$

where  $T_{n,m}^1(q)$  and  $T_{n,m}^2(q)$  can be simplified as follows:

$$\begin{aligned}
T_{n,m}^2(q) &= \left( 1 - \frac{q^{m-1} [n]_q}{[n+m-1]_q} \right) \frac{1}{[n+1]_q} \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right) \\
&= \frac{[m-1]_q}{[n+m-1]_q [n+1]_q} \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right)
\end{aligned}$$

and



$$\begin{aligned}
T_{n,m}^1(q) &= \frac{[m-1]_q}{[n+m-1]_q} + \frac{q^{m-1}[m-1]_q}{[n+m-1]_q} \\
&\quad - \frac{q^{m-1}[n]_q}{[n+m-1]_q} \frac{1}{[n+1]_q} \left( q[m-1]_q + [m-1]_{q^{-1}} \right) \\
&= [m-1]_q \left( \frac{1}{[n+m-1]_q} - \frac{q}{[n+1]_q} \frac{q^{m-1}[n]_q}{[n+m-1]_q} \right) \\
&\quad + [m-1]_q \left( \frac{q^{m-1}}{[n+m-1]_q} - \frac{1}{[n+1]_q} \frac{q[n]_q}{[n+m-1]_q} \right) \\
&= [m-1]_q \frac{[n+1]_q - q^m[n]_q}{[n+m-1]_q[n+1]_q} + [m-1]_q \frac{q^{m-1}[n+1]_q - q[n]_q}{[n+m-1]_q[n+1]_q} \\
&= [m-1]_q \frac{(1+q^{m-1})[n+1]_q - (1+q^{m-1})q[n]_q}{[n+m-1]_q[n+1]_q} \\
&= \frac{[m-1]_q(1+q^{m-1})}{[n+m-1]_q[n+1]_q}.
\end{aligned}$$

■

**Lemma 12** *Let  $q > 1$  and  $f \in H(\mathbb{D}_R)$ . The function  $L_q(f; z)$  has the following representation*

$$L_q(f; z) = \sum_{m=2}^{\infty} a_m \left( q \sum_{i=1}^{m-1} [i] + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) z^{m-1} (1-z), \quad z \in \mathbb{D}_R.$$

**Proof.** Using the following identity

$$\begin{aligned}
[m]_q - m &= 1 + q + q^2 + \dots + q^{m-1} - m \\
&= (1-1) + (q-1) + (q^2-1) + \dots + (q^{m-1}-1) \\
&= (q-1)[1]_q + (q-1)[2]_q + \dots + (q-1)[m-1]_q + \\
&= (q-1) \left( [1]_q + \dots + [m-1]_q \right) = (q-1) \sum_{i=1}^{m-1} [i]_q,
\end{aligned}$$

we get

$$\begin{aligned}
L_q(f; z) &= \sum_{m=2}^{\infty} a_m \left( \frac{q([m]_q - [m]_{q^{-1}})}{q-1} \right) z^{m-1} (1-z) \\
&= \sum_{m=2}^{\infty} a_m \left( \frac{q([m]_q - m)}{q-1} + \frac{[m]_{q^{-1}} - m}{q^{-1} - 1} \right) z^{m-1} (1-z) \\
&= \sum_{m=2}^{\infty} a_m \left( q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) z^{m-1} (1-z),
\end{aligned}$$

where  $f(z) = \sum_{m=0}^{\infty} a_m z^m$ . ■

## 4 Proofs of the main results

Firstly we prove that  $U_{n,q}(f; z) = \sum_{m=0}^{\infty} a_m U_{n,q}(e_m, z)$ . Indeed denoting  $f_k(z) = \sum_{j=0}^k a_j z^j, |z| \leq r$  with  $m \in \mathbb{N}$ , by the linearity of  $U_{n,q}$ , we have

$$U_{n,q}(f_k, z) = \sum_{m=0}^k a_m U_{n,q}(e_m, z),$$

and it is sufficient to show that for any fixed  $n \in \mathbb{N}$  and  $|z| \leq r$  with  $r \geq 1$ , we have  $\lim_{k \rightarrow \infty} U_{n,q}(f_k, z) = U_{n,q}(f; z)$ . But this is immediate from  $\lim_{k \rightarrow \infty} \|f_k - f\|_r = 0$ , the norm being the defined as  $\|f\|_r = \max\{|f(z)| : |z| \leq r\}$  and from the inequality

$$\begin{aligned} & |U_{n,q}(f_k, z) - U_{n,q}(f, z)| \\ & \leq |f_k(0) - f(0)| \cdot |(1-z)^n| + |f_k(1) - f(1)| \cdot |z^n| \\ & + [n+1]_{q^{-1}} \sum_{j=1}^{n-1} |p_{n,j}(q; z)| q^{j-1} \int_0^1 p_{n-2,j-1}(q^{-1}, q^{-1}t) |f_k(t) - f(t)| d_{q^{-1}} t \\ & \leq C_{r,n} \|f_k - f\|_r, \end{aligned}$$

valid for all  $|z| \leq r$ , where

$$\begin{aligned} C_{r,n} &= (1+r)^n + r^n + [n+1]_{q^{-1}} \sum_{j=1}^{n-1} \left[ \begin{matrix} n \\ j \end{matrix} \right]_q (1+r)^{n-j} r^j q^{j-1} \int_0^1 p_{n-2,j-1}(q^{-1}; q^{-1}t) d_{q^{-1}} t \\ &= (1+r)^n + r^n + \sum_{j=1}^{n-1} \left[ \begin{matrix} n \\ j \end{matrix} \right]_q (1+r)^{n-j} r^j q^{j-1}. \end{aligned}$$

Therefore we get

$$|U_{n,q}(f; z) - f(z)| \leq \sum_{m=0}^{\infty} |a_m| |U_{n,q}(e_m, z) - e_m(z)| = \sum_{m=2}^{\infty} |a_m| |U_{n,q}(e_m, z) - e_m(z)|,$$

as  $U_{n,q}(e_0, z) = e_0(z)$  and  $U_{n,q}(e_1, z) = e_1(z)$ .

**Proof of Theorem 2.** From the recurrence formula (8) and the inequality (5) for  $m \geq 2$  we get

$$\begin{aligned} |U_{n,q}(e_m; z) - z^m| &\leq \frac{q^{m-1} z (1-z)}{q^{m-2} [n+1]_q + [m-2]_q} |D_q U_{n,q}(e_{m-1}; z)| \\ &+ \frac{q^{m-1} [n]_q z + [m-1]_q}{q^{m-1} [n]_q + [m-1]_q} |U_{n,q}(e_{m-1}; z) - z^{m-1}| + \frac{[m-1]_q}{q^{m-2} [n+1]_q + [m-2]_q} |1-z| |z|^{m-1}. \end{aligned}$$

It is known that by a linear transformation, the Bernstein inequality in the closed unit disk becomes

$$|P'_k(z)| \leq \frac{k}{qr_1} \|P_k\|_{qr}, \quad \text{for all } |z| \leq qr, \quad r \geq 1,$$

which combined with the mean value theorem in complex analysis implies

$$|D_q(P_k; z)| \leq \|P'_k\|_{qr} \leq \frac{k}{qr} \|P_k\|_{qr},$$

for all  $|z| \leq qr$ , where  $P_k(z)$  is a complex polynomial of degree  $\leq k$ . It follows that

$$\begin{aligned}
& |U_{n,q}(e_m; z) - z^m| \\
& \leq \frac{q^{m-1}r(1+r)}{q^{m-2}[n+1]_q + [m-2]_q} \frac{m-1}{qr} \|U_{n,q}(e_{m-1})\|_{qr} \\
& + r |U_{n,q}(e_{m-1}; z) - z^{m-1}| + \frac{[m-1]_{1/q}}{[n+1]_q} (1+r) r^{m-1} \\
& \leq \frac{(m-1)}{[n+1]_q} (1+r) q^{m-1} r^{m-1} + r |U_{n,q}(e_{m-1}; z) - z^{m-1}| + \frac{[m-1]_{1/q}}{[n+1]_q} (1+r) r^{m-1} \\
& \leq 2q(m-1) \frac{r(1+r)}{[n+1]_q} (qr)^{m-2} + r |U_{n,q}(e_{m-1}; z) - z^{m-1}|.
\end{aligned}$$

By writing the last inequality for  $m = 2, 3, \dots$ , we easily obtain, step by step the following

$$\begin{aligned}
|U_{n,q}(e_m; z) - z^m| & \leq r \left( r |U_{n,q}(e_{m-2}; z) - z^{m-2}| + 2 \frac{(m-2)}{[n+1]_q} r(1+r) (qr)^{m-3} \right) \\
& + 2 \frac{(m-1)}{[n+1]_q} r(1+r) (qr)^{m-2} \\
& = r^2 |U_{n,q}(e_{m-2}; z) - z^{m-2}| + 2 \frac{r(1+r)}{[n+1]_q} r^{m-2} (m-1 + m-2) \\
& \leq \dots \leq \frac{r(1+r)}{[n+1]_q} m(m-1) q^{m-2} r^{m-2}.
\end{aligned}$$

It follows that

$$|U_{n,q}(f; z) - f(z)| \leq \sum_{m=2}^{\infty} |a_m| |U_{n,q}(e_m; z) - z^m| \leq \frac{r(1+r)}{[n+1]_q} \sum_{m=2}^{\infty} |a_m| m(m-1) q^{m-2} r^{m-2}.$$

■

The second main result of the paper is the Voronovskaja theorem with a quantitative estimate for the complex version of genuine  $q$ -Bernstein-Durrmeyer polynomials.

**Proof of Theorem 3.** By Lemma 11 we have

$$\begin{aligned}
\Theta_{n,m}(q; z) & = \frac{q^{m-1}z(1-z)}{[n+m-1]_q} D_q(U_{n,q}(e_{m-1}; z) - z^{m-1}) \\
& + \frac{q^{m-1}[n]z + [m-1]_q}{[n+m-1]_q} \Theta_{n,m-1}(q; z) + R_{n,m}(q; z),
\end{aligned} \tag{11}$$

where

$$R_{n,m}(q; z) = \frac{[m-1]_q}{[n+m-1]_q [n+1]_q} \left[ (1+q^{m-1}) + \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right) (z+1) \right] z^{m-2} (1-z).$$

It follows that

$$\begin{aligned}
|R_{n,m}(q; z)| & \leq \frac{[m-1]_q}{[n+1]_q^2} \left( (1+q^{m-1})r + \left( q \sum_{i=1}^{m-2} [i]_q + \sum_{i=1}^{m-2} [i]_{q^{-1}} \right) (1+r) \right) (1+r) r^{m-2} \\
& \leq \frac{[m-1]_q}{[n+1]_q^2} \left( (1+q^{m-1}) + \left( q(m-2)[m-2]_q + (m-2)^2 \right) \right) (1+r)^2 r^{m-2} \\
& = \frac{q^{m-2}[m-1]_{q^{-1}}}{[n+1]_q^2} q^{m-2} \left( \left( \frac{1}{q^{m-2}} + q \right) + (m-2)[m-2]_{q^{-1}} + \frac{1}{q^{m-2}} (m-2)^2 \right) (1+r)^2 r^{m-2} \\
& \leq \frac{3}{[n+1]_q^2} (m-1)(m-2)^2 (1+r)^2 (q^2 r)^{m-2}
\end{aligned}$$

for all  $m \geq 2$ ,  $n \in \mathbb{N}$  and  $z \in \mathbb{C}$ . (11) implies that for  $|z| \leq r$

$$\begin{aligned}
|\Theta_{n,m}(q; z)| &\leq r |\Theta_{n,m-1}(q; z)| + \frac{q^{m-1}r(1+r)}{q^{m-2}[n+1]_q} \frac{m-1}{qr} \|U_{n,q}(e_{m-1}) - e_{m-1}\|_{qr} \\
&\quad + \frac{3}{[n+1]_q^2} (m-1)(m-2)^2 (1+r)^2 (q^2 r)^{m-2} \\
&\leq r |\Theta_{n,m-1}(q; z)| + \frac{r^2(1+r)^2}{[n+1]_q^2} (m-1)^2 (m-2) (q^2 r)^{m-3} \\
&\quad + \frac{3}{[n+1]_q^2} (m-1)(m-2)^2 (1+r)^2 (q^2 r)^{m-2} \\
&\leq r |\Theta_{n,m-1}(q; z)| + \frac{4r^2(1+r)^2}{[n+1]_q^2} (m-1)^2 (m-2) (q^2 r)^{m-2}.
\end{aligned}$$

By writing the last inequality for  $m = 3, 4, \dots$ , we easily obtain, step by step the following

$$\begin{aligned}
&\left| U_{n,q}(f; z) - f(z) - \frac{1}{[n+1]_q} L_q(f; z) \right| \\
&\leq \frac{4r^2(1+r)^2}{[n+1]_q^2} \sum_{m=2}^{\infty} |a_m| (q^2 r)^{m-2} \sum_{j=2}^m (j-1)^2 (j-2) \leq \frac{4r^2(1+r)^2}{[n+1]_q^2} \sum_{m=2}^{\infty} |a_m| (m-1)^2 (m-2)^2 (q^2 r)^{m-2}.
\end{aligned}$$

■

**Proof of Theorem 4.** For all  $z \in \mathbb{D}_R$  and  $n \in \mathbb{N}$  we get

$$U_{n,q}(f; z) - f(z) = \frac{1}{[n+1]_q} \left\{ L_q(f; z) + [n+1]_q \left( U_{n,q}(f; z) - f(z) - \frac{1}{[n+1]_q} L_q(f; z) \right) \right\}.$$

It follows that

$$\|U_{n,q}(f) - f\|_r \geq \frac{1}{[n+1]_q} \left\{ \|L_q(f; z)\|_r - [n+1]_q \left\| U_{n,q}(f) - f - \frac{1}{[n+1]_q} L_q(f; z) \right\|_r \right\}.$$

Because by hypothesis  $f$  is not a polynomial of degree  $\leq 1$  in  $\mathbb{D}_R$ , it follows  $\|L_q(f; z)\|_r > 0$ . Indeed, assuming the contrary it follows that  $L_q(f; z) = 0$  for all  $z \in \mathbb{D}_r$  that is  $D_q f(z) = D_{q^{-1}} f(z)$  for all  $z \in \mathbb{D}_r$ . Thus  $a_m = 0$ ,  $m = 2, 3, \dots$  and,  $f$  is linear, which is contradiction with the hypothesis.

Now, by Theorem 3 we have

$$\begin{aligned}
[n+1]_q \left| U_{n,q}(f; z) - f(z) - \frac{1}{[n+1]_q} L_q(f; z) \right| &\leq \frac{4r^2(1+r)^2}{[n+1]_q} \sum_{m=3}^{\infty} |a_m| (m-1)^2 (m-2)^2 (q^2 r)^{m-2} \\
&\rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Consequently, there exists  $n_1$  (depending only on  $f$  and  $r$ ) such that for all  $n \geq n_1$  we have

$$\|L_q(f; z)\|_r - [n+1]_q \left\| U_{n,q}(f) - f - \frac{1}{[n+1]_q} L_q(f; z) \right\|_r \geq \frac{1}{2} \|L_q(f; z)\|_r,$$

which implies

$$\|U_{n,q}(f) - f\|_r \geq \frac{1}{2[n+1]_q} \|L_q(f; z)\|_r, \quad \text{for all } n \geq n_1.$$

For  $1 \leq n \leq n_1 - 1$  we have

$$\|U_{n,q}(f) - f\|_r \geq \frac{1}{[n+1]_q} \left( [n+1]_q \|U_{n,q}(f) - f\|_r \right) = \frac{1}{[n+1]_q} M_{r,n,q}(f) > 0,$$

which finally implies that

$$\|U_{n,q}(f) - f\|_r \geq \frac{1}{[n+1]_q} C_{r,q}(f),$$

for all  $n$ , with  $C_{r,q}(f) = \min \{M_{r,1,q}(f), \dots, M_{r,n_1-1,q}(f), \frac{1}{2} \|L_q(f; z)\|_r\}$ , which ends the proof. ■

**Proof of Theorem 6.** Let  $1 \leq r < R$ , let  $1 < q_0 < \frac{R}{r}$  be fixed. Then by Lemma 12 for any  $1 \leq q \leq q_0$  and  $|z| \leq r$ , we have

$$\begin{aligned} L_q(f; z) &= \sum_{m=2}^{\infty} a_m \left( q \sum_{i=1}^{m-1} [i]_q + \sum_{i=1}^{m-1} [i]_{q^{-1}} \right) z^{m-1} (1-z), \\ L_1(f; z) &= \sum_{m=2}^{\infty} a_m m(m-1) z^{m-1} (1-z). \end{aligned}$$

Using the inequality

$$\begin{aligned} \left| q \sum_{i=1}^{m-1} [i]_q - \frac{m(m-1)}{2} \right| &= q \sum_{i=2}^{m-1} \left( [i]_q - i \right) + (q-1) \frac{m(m-1)}{2} \\ &= q(q-1) \sum_{i=2}^{m-1} \sum_{j=1}^i [j]_q + (q-1) \frac{m(m-1)}{2} \\ &\leq q(q-1) [m-1]_q \frac{m(m-1)}{2} + (q-1) \frac{m(m-1)}{2} \\ &= (q-1) \frac{m(m-1)}{2} (q[m-1]_q + 1) \\ &\leq (q-1) q^{m-1} \frac{m^2(m-1)}{2} \end{aligned}$$

and

$$\begin{aligned} \left| \sum_{i=1}^{m-1} [i]_{q^{-1}} - \frac{m(m-1)}{2} \right| &= \sum_{i=2}^{m-1} (i - [i]_{q^{-1}}) \\ &= (1 - q^{-1}) \sum_{i=2}^{m-1} \sum_{j=1}^i [j]_{q^{-1}} \\ &\leq (1 - q^{-1}) \frac{m(m-1)^2}{2}, \end{aligned}$$

we get for  $1 \leq q \leq q_0$  and  $|z| \leq r$ ,

$$\begin{aligned} &|L_q(f; z) - L_1(f; z)| \\ &\leq \sum_{m=2}^{N-1} |a_m| \left| q \sum_{i=1}^{m-1} [i]_q - \frac{m(m-1)}{2} \right| |z^{m-1} - z^m| + \sum_{m=N}^{\infty} |a_m| \left| q \sum_{i=1}^{m-1} [i]_q - \frac{m(m-1)}{2} \right| |z^{m-1} - z^m| \\ &+ \sum_{m=2}^{N-1} |a_m| \left| \sum_{i=1}^{m-1} [i]_{q^{-1}} - \frac{m(m-1)}{2} \right| |z^{m-1} - z^m| + \sum_{m=N}^{\infty} |a_m| \left| \sum_{i=1}^{m-1} [i]_{q^{-1}} - \frac{m(m-1)}{2} \right| |z^{m-1} - z^m| \\ &\leq (q-1) \sum_{m=2}^{N-1} |a_m| m^2(m-1) q_0^{m-1} r^m + 4 \sum_{m=N}^{\infty} |a_m| (m-1)^2 q_0^m r^m \\ &+ (1 - q^{-1}) \sum_{m=2}^{N-1} |a_m| m(m-1)^2 r^m + 2 \sum_{m=N}^{\infty} |a_m| m(m-1) r^m. \end{aligned}$$

Since  $f \in H(\mathbb{D}_R)$ , we can find  $N = N_\varepsilon \in \mathbb{N}$  such that

$$4 \sum_{m=N}^{\infty} |a_m| (m-1)^2 q_0^m r^m + 2 \sum_{m=N}^{\infty} |a_m| m(m-1) r^m < \varepsilon/2.$$

Thus for  $q$  sufficiently close to 1 from the right, we conclude that

$$\lim_{q \rightarrow 1^+} L_q(f; z) = L_1(f; z)$$

uniformly on  $\mathbb{D}_r$ . The proof is finished. ■

**Proof of Theorem 5.** Then by Theorem 3, we get  $L_q(f; z) = \lim_{n \rightarrow \infty} [n+1]_q (U_{n,q}(f; z) - f(z)) = 0$  for infinite number of points having an accumulation point on  $\mathbb{D}_{R/q}$ . Since  $L_q(f; z) \in H(\mathbb{D}_{R/q})$ , by the Unicity Theorem for analytic functions we get  $L_q(f; z) = 0$  in  $\mathbb{D}_{R/q}$ , and therefore, by (2),  $a_m = 0$ ,  $m = 2, 3, \dots$ . Thus,  $f$  is linear. Theorem 5 is proved. ■

## References

- [1] Andrews G E, Askey R, Roy R. Special functions. Cambridge: Cambridge University Press; 1999.
- [2] Majid S. Foundations of quantum group theory. Cambridge: Cambridge University Press; 2000.
- [3] Phillips, G. M. A survey of results on the  $q$ -Bernstein polynomials. IMA J. Numer. Anal. 30 (2010), no. 1, 277–288.
- [4] S.G. Gal, Approximation by complex genuine Durrmeyer type polynomials in compact disks. Appl. Math. Comput. 217 (2010), no. 5, 1913–1920.
- [5] P. E. Parvanov & B.D. Popov, The limit case of Bernstein’s operators with Jacobi weights. Math. Balkanica (N. S.) 8 (1994), 165–177.
- [6] G. G. Lorentz, Bernstein Polynomials (Chelsea, New York, 1986).
- [7] T. Sauer, The genuine Bernstein–Durrmeyer operator on a simplex. Result. Math. 26 (1994), 99–130.
- [8] S. Waldron, A generalised beta integral and the limit of the Bernstein–Durrmeyer operator with Jacobi weights. J. Approx. Theory 122 (2003), 141–150.
- [9] R. Páltanea, Approximation Theory using Positive Linear Operators. Boston: Birkhäuser 2004.
- [10] Chen W., On the modified Bernstein-Durrmeyer operator. In: Report of the Fifth Chinese Conference on Approximation Theory, Zhen Zhou, China, 1987.
- [11] Derriennic M-M., Modified Bernstein polynomials and Jacobi polynomials in  $q$ -calculus. Rendiconti Del Circolo Matematico Di Palermo, Serie II 2005; 76(Suppl.):269–290.
- [12] DeVore, R.A. and Lorentz, G.G., Constructive Approximation, Springer, Berlin, 1993.
- [13] Gonska, H., Kacsó, D. and Raşa, I., On Genuine Bernstein–Durrmeyer Operators, Result.Math. 50 (2007), 213–225.
- [14] Gonska, H., Piţul, P., Raşa, I.: On Peano’s form of the Taylor remainder, Voronovskaja’s theorem and the commutator of positive linear operators. In: Proceed. Intern. Conf. on “Numer. Anal., Approx. Theory”, NAAT, Cluj-Napoca, Casa Cartii de Stiinta, Cluj-Napoca, pp. 55–80, 2006
- [15] Goodman, T. N. T. and Sharma, A., A Bernstein type operator on the simplex, Math. Balkanica 5 (1991), 129–145.
- [16] G. A. Anastassiou and S.G. Gal, Approximation by Complex Bernstein-Durrmeyer Polynomials in Compact Disks, *Mediterr. J. Math.*, 7(2010), No. 4, 471–482.

- [17] S. Ostrovska,  $q$ -Bernstein polynomials and their iterates. J. Approx. Theory 123 (2003), 232–255.
- [18] S.Ostrovska: The sharpness of convergence results for  $q$ -Bernstein polynomials in the case  $q > 1$ , Czechoslovak Mathematical Journal, 58 (133) (2008), 1195–1206.
- [19] Mahmudov N.I., Sabancıgil P., On genuine  $q$ -Bernstein–Durrmeyer operators, Publ. Math. Debrecen, 76 (2010), no. 3-4, 465–479.
- [20] N.I. Mahmudov, Convergence properties and iterations for  $q$ -Stancu polynomials in compact disks, Computer and Mathematics with Applications, Volume 59, Issue 12, June 2010, Pages 3763-3769.
- [21] Wang, H. and Wu, X., Saturation of convergence of  $q$ -Bernstein polynomials in the case  $q > 1$ . J. Math. Anal. Appl., 2008; 337(1):744–750.
- [22] Z.Wu, The saturation of convergence on the interval  $[0; 1]$  for the  $q$ -Bernstein polynomials in the case  $q > 1$ ; J. Math. Anal. Appl., Volume 357, Issue 1, 2009, Pages 137-141.
- [23] S. G. Gal, Approximation by complex Bernstein-Durrmeyer polynomials with Jacobi weights in compact disks, *Mathematica Balkanica (N.S.)*, **24**(2010), no. 1-2, 103–119.
- [24] S. G. Gal, Approximation by complex genuine Durrmeyer type polynomials in compact disks, *Appl. Math. Comput.*, **217**(2010), 1913–1920.
- [25] N. I. Mahmudov, Approximation by genuine  $q$  -Bernstein-Durrmeyer polynomials in compact disks, *Haceteepe Journal of Mathematics and Statistics*, **40** (1) (2011), 77-89.
- [26] N. I. Mahmudov, Approximation by Bernstein–Durrmeyer-type operators in compact disks, *Applied Mathematics Letters*, **24** (7) (2011), 1231-1238.
- [27] N. I. Mahmudov, The moments for  $q$ -Bernstein operators in the case  $0 < q < 1$ ,. Numer. Algorithms 53 (2010), no. 4, 439–450.
- [28] S. G. Gal, V. Gupta, N. I. Mahmudov, Approximation by a complex  $q$ -Durrmeyer type operator, Ann Univ Ferrara (2012) 58:65–87.
- [29] S. G. Gal, Overconvergence in Complex Approximation, Springer New York Heidelberg Dordrecht London, 2013.